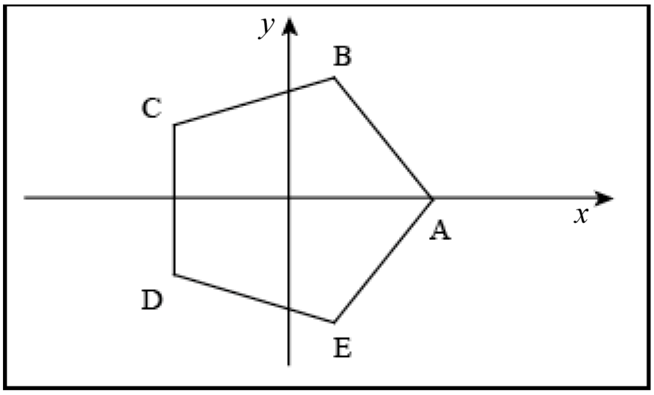
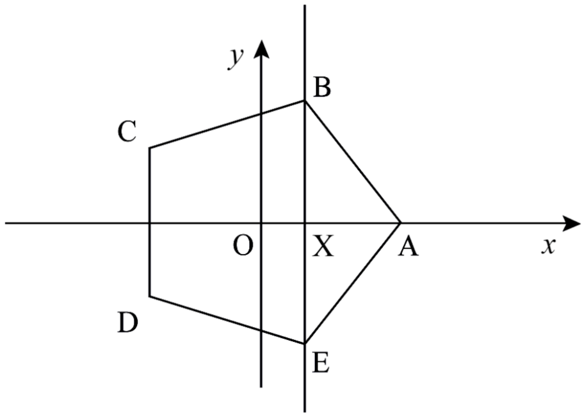
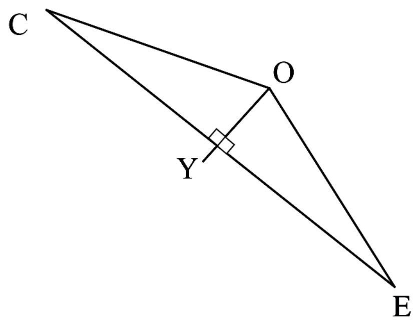
Assessment Schedule – 2005

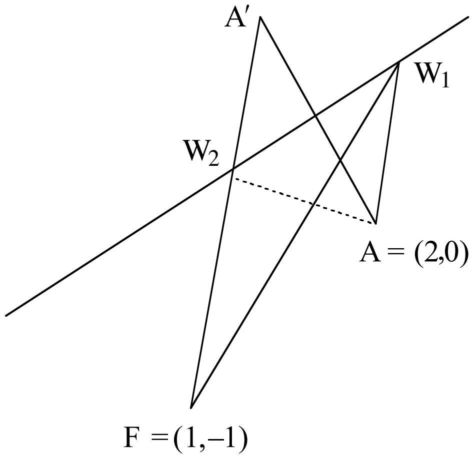
Scholarship Mathematics with Calculus: (93202)

Evidence Statement

As shown in the schedule below, either a seven-point marking scale (0–6), or a nine-point marking scale (0–8), was used to assess the questions.

Question	Evidence	Code	Judgement
ONE (a)	$z^5 - 1 = 0$ $\alpha^5 - 1 = 0$ $(\alpha - 1)(\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1) = 0$ but α is complex so $\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$ $\alpha^4 + \alpha^3 + \alpha^2 + \alpha = -1.$ Sum of roots is $\alpha^4 + \alpha^3 + \alpha^2 + \alpha = -1$ from above.	**6** **2**	
	Product of roots is $(\alpha + \alpha^4)(\alpha^2 + \alpha^3) = \alpha^3 + \alpha^4 + \alpha^6 + \alpha^7$ But $\alpha^5 = 1$ $\alpha^3 + \alpha^4 + \alpha^6 + \alpha^7 = \alpha^3 + \alpha^4 + \alpha + \alpha^2 = -1$ hence the equation is: $z^2 + z - 1 = 0.$		No simplification: –1 mark. No 'hence': –1 mark. no sum of roots: max. 4 marks.
ONE (b)(i)	 **Either** As shown following, BE is perpendicular to the *x*-axis (by symmetry, congruent triangles, AXB and AXE, **SAS**).	**6**	

Question	Evidence	Code	Judgement
ONE (b)(i) contd	 So BE = $2b$, but BE = CE (triangles ABE and DEC congruent **SAS**) So CE = $2b$. **Or** $\angle BOA = \frac{2\pi}{5}$ (72°) and OB = 2, hence $b = 2\sin\frac{2\pi}{5}$. Triangle COE is isosceles (CO = EO = 2).  So CY = EY = $2\sin\frac{2\pi}{5} = b$ and so CE = $2b$. Alternative for this last step is the cosine rule: $CE^2 = 2^2 + 2^2 - 2 \bullet 2 \bullet 2\cos\frac{4\pi}{5}$ $CE^2 = 8\left(1 - \cos\frac{4\pi}{5}\right)$ $= 8\left(1 - (1 - 2\sin^2\frac{2\pi}{5})\right) = 8\left(2\sin^2\frac{2\pi}{5}\right) = 16\sin^2\frac{2\pi}{5}$ and CE $= 4\sin\frac{2\pi}{5} = 2b$.		Accept degrees. Answer must be in terms of b. Accept decimals if = $2b$. If answer from decimals is only inferred: –1 mark.

Question	Evidence	Code	Judgement
ONE (b)(ii)	 **Either** Using calculus, by Pythagoras' theorem, and W = (t,t) $\mathrm{AW} = \sqrt{(2-t)^2 + t^2} = \sqrt{4-4t+2t^2}$ $\mathrm{FW} = \sqrt{(t-1)^2 + (t+1)^2} = \sqrt{2t^2+2} = \sqrt{2}\sqrt{t^2+1}$ $\mathrm{AW} + \mathrm{FW} = \sqrt{4-4t+2t^2} + \sqrt{2}\sqrt{t^2+1}$ $\frac{\mathrm{d}(\mathrm{AW}+\mathrm{FW})}{\mathrm{d}t} = \frac{2(t-1)}{\sqrt{4-4t+2t^2}} + \sqrt{2}\frac{t}{\sqrt{t^2+1}}$ so for a max. / min. $\frac{\mathrm{d}(\mathrm{AW}+\mathrm{FW})}{\mathrm{d}t} = 0$ and so $\frac{2(t-1)}{\sqrt{4-4t+2t^2}} + \sqrt{2}\frac{t}{\sqrt{t^2+1}} = 0$	8 6	
	$4(t-1)^2(t^2+1) = 2t^2(4-4t+2t^2) = 4t^2(2-2t+t^2)$ $(t^2-2t+1)(t^2+1) = t^2(2-2t+t^2)$ $t^4 - 2t^3 + 2t^2 - 2t + 1 = 2t^2 - 2t^3 + t^4$ $-2t+1=0, \quad t = \frac{1}{2}.$ $\mathrm{AW} + \mathrm{FW} = \sqrt{4 - 4\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^2} + \sqrt{2}\sqrt{\left(\frac{1}{2}\right)^2 + 1}$		

Question	Evidence	Code	Judgement
ONE (b)(ii) contd	$= \sqrt{2\frac{1}{2}} + \sqrt{2}\sqrt{1\frac{1}{4}} = \sqrt{2\frac{1}{2}} + \sqrt{2\frac{1}{2}} = 2\sqrt{\frac{5}{2}} = \sqrt{10}.$ **Or:** Geometrically, let A′ be the reflection of A in the line $y = x$ that W lies on. So A′ = (0,2). Then AW = A′W and the minimum value of A′W + FW is when A′F is a straight line (ie for the point W_2 as shown above). Here $A'W_2 + FW_2 = A'F$ and by Pythagoras' theorem $A'F = \sqrt{1^2 + 3^2} = \sqrt{10}$, the minimum value. **Or:** Gradient of AF = gradient of the line $y = x$ that W lies on. So they are parallel. Hence the minimum distance is when WAF is an isosceles triangle, and the line from W is perpendicular to AF. Hence AW + FW = 2AW = 2FW Equation of AF is $y = x - 2$ and perpendicular distance between the lines = $2 \sin 45^\circ = \sqrt{2}$ $AF^2 = 1^2 + (-1)^2 = 2$ so $AW^2 = \left(\sqrt{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2$ and minimum AW + FW = 2AW = $2\sqrt{\frac{5}{2}} = \sqrt{10}$.		Accept 3.162 or decimal equivalent.

<table>
<tr><td>TWO
(a)</td><td>

The equation of the circle $(x-a)^2+(y-b)^2=r^2$

The point $\mathrm{A}(x,y)\to\mathrm{B}(x',y')$ A, B correct

$x'=x \qquad x=x'$

$y'=hy \qquad y=\dfrac{y'}{h}$

substituting gives

$$(x'-a)^2+\left(\frac{y'}{h}-b\right)^2=r^2$$

$$\frac{(x'-a)^2}{r^2}+\frac{(y'-hb)^2}{h^2r^2}=1$$

$h=\frac{1}{2}$ equation of ellipse is $\dfrac{(x-a)^2}{r^2}+\dfrac{\left(y-\frac{1}{2}b\right)^2}{\frac{r^2}{4}}=1$

Method 1

$(x-a)^2+(2y-b)^2=r^2$

on the y-axis $x=0$ $\quad a^2+(2y-b)^2=r^2$

$$2y-b=\pm\sqrt{r^2-a^2}$$

$$y=\frac{b\pm\sqrt{r^2-a^2}}{2}$$

The circle cuts the y-axis when

$a^2+(y-b)^2=r^2$

$y-b=\pm\sqrt{r^2-a^2}$

$y=b\pm\sqrt{r^2-a^2}$

For the ellipse and the circle to intersect on the y-axis:

$$\frac{b+\sqrt{r^2-a^2}}{2}=b-\sqrt{r^2-a^2}$$

</td><td>6

4</td><td></td></tr>
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$$b+\sqrt{r^2-a^2}=2b-2\sqrt{r^2-a^2}$$

$$b=3\sqrt{r^2-a^2}$$

$$b^2=9\left(r^2-a^2\right)$$

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TWO (a) contd	**Method 2** Solve equations simultaneously: $(x-a)^2+(2y-b)^2=(x-a)^2+(y-b)^2$ $2y-b=\pm(y-b)$ $2y-b=+y-b$ or $2y-b=-(y-b)$ $y=0$ $\quad$ $y-b=-y+b$ or $y=\frac{2b}{3}$ To meet on the *y*-axis $x=0$, so for the circle using $y=\frac{2b}{3}$ $(x-a)^2+(y-b)^2=r^2$ $a^2+\left(\frac{2b}{3}-b\right)^2=r^2$	**4**	
	$\left(\frac{b}{3}\right)^2=r^2-a^2$ $b^2=9\left(r^2-a^2\right)$ **Method 3** If the *y* co-ordinate of one point where the ellipse and the circle intersect on the *y*-axis is y_1 then the other is $0.5y_1$ For the circle, $(x-a)^2+(y-b)^2=r^2$, when $x=0$, $y^2-2by+b^2+a^2-r^2=0$ but the sum of these roots is $1.5y_1$ and the product $0.5{y_1}^2$ so $\frac{3}{2}y_1=2b$ $\frac{1}{2}{y_1}^2=b^2+a^2-r^2$ and so $\frac{1}{2}\left(\frac{16}{9}b^2\right)=b^2+a^2-r^2$, $\frac{1}{9}b^2=r^2-a^2$, $b^2=9\left(r^2-a^2\right)$.		
TWO (b)	Circle: $x^2+y^2=r^2$ $y^2=r^2-x^2$ $y=\pm\sqrt{r^2-x^2}$ Since we require the top half of the circle, $y=+\sqrt{r^2-x^2}$	**6**	

TWO (b) contd	Ellipse: $x^2 + 16(y-r)^2 = r^2$ **Method 1** $x^2 + y^2 = r^2$ $\therefore$ Meet where (subtracting) $16(y-r)^2 - y^2 = 0$ $4(y-r) = \pm y$ $5y = 4r$ or $3y = 4r$ $y = \frac{4r}{5}$ or $\frac{4r}{3}$ but $y < r$ $\therefore y = \frac{4r}{5}$ $\therefore x^2 = r^2 - \frac{16r^2}{25} = \frac{9r^2}{25}$ $x = \pm\frac{3r}{5}$	**2**	
	Method 2 $(y-r)^2 = \frac{r^2 - x^2}{16}$ $y - r = \pm\sqrt{\frac{r^2 - x^2}{16}}$ $y = r \pm \frac{\sqrt{r^2 - x^2}}{4}$ For $y < r$ $y = r - \frac{\sqrt{r^2 - x^2}}{4}$ Solving for points of intersection:		

TWO (b) Contd	$$r - \frac{\sqrt{r^2 - x^2}}{4} = \sqrt{r^2 - x^2}$$ $$\sqrt{r^2 - x^2}\left(1 + \frac{1}{4}\right) = r$$ $$\sqrt{r^2 - x^2} = \frac{4r}{5}$$ $$r^2 - x^2 = \frac{16r^2}{25}$$ $$x^2 = \left(1 - \frac{16}{25}\right)r^2$$ $$= \frac{9r^2}{25}$$ $$x = \pm\frac{3r}{5}$$	**2**	
	$$\text{Area} = 2\int_0^{\frac{3r}{5}} \sqrt{r^2 - x^2} - \left(r - \frac{\sqrt{r^2 - x^2}}{4}\right) \mathrm{d}x$$ $$= 2\int_0^{\frac{3r}{5}} \frac{5}{4}\sqrt{r^2 - x^2} - r\,\mathrm{d}x$$ $$= 2\int_0^{\frac{3r}{5}} \frac{5}{4}\sqrt{r^2 - x^2}\,\mathrm{d}x \; - 2\int_0^{\frac{3r}{5}} r\,\mathrm{d}x$$ using the substitution: $x = r\sin u \quad x = 0 \quad u = 0$ $\mathrm{d}x = r\cos u\,\mathrm{d}u \quad x = \frac{3r}{5} \quad u = \sin^{-1}\left(\frac{3}{5}\right) = 0.6435$ $$2\int_0^{\frac{3r}{5}} \frac{5}{4}\sqrt{r^2 - x^2}\,\mathrm{d}x = 2\int_0^{\sin^{-1}\left(\frac{3}{5}\right)} \frac{5}{4}\sqrt{(r^2 - r^2\sin^2 u)}\; r\cos u\,\mathrm{d}u$$	**4**	Accept decimal limit.
	$$= 2\int_0^{\sin^{-1}\left(\frac{3}{5}\right)} \frac{5}{4} r\cos u\; r\cos u\,\mathrm{d}u$$ $$= 2 \times \frac{5}{4} r^2 \int_0^{\sin^{-1}\left(\frac{3}{5}\right)} \cos^2 u\,\mathrm{d}u$$ $$= \frac{5}{4} r^2 \int_0^{\sin^{-1}\left(\frac{3}{5}\right)} \left(\cos 2u + 1\right)\mathrm{d}u$$		

	$= \frac{5}{4}r^2\left[\frac{1}{2}\sin 2u + u\right]_0^{\sin^{-1}\left(\frac{3}{5}\right)}$ $= \frac{5}{4}r^2\left[\sin u \cos u + u\right]_0^{\sin^{-1}\left(\frac{3}{5}\right)}$ $= \frac{5}{4}r^2\frac{3}{5}\bullet\frac{4}{5} + \frac{5}{4}r^2\sin^{-1}\left(\frac{3}{5}\right)$ $= \frac{3}{5}r^2 + \frac{5}{4}r^2\sin^{-1}\left(\frac{3}{5}\right)$ $\int_0^{\frac{3r}{5}} 2r\,\mathrm{d}x = \left[2rx\right]_0^{\frac{3r}{5}} = \frac{6r^2}{5}$ $\text{Area} = \frac{3}{5}r^2 + \frac{5}{4}r^2\sin^{-1}\left(\frac{3}{5}\right) - \frac{6}{5}r^2$ $\text{Area} = \frac{5}{4}r^2\sin^{-1}\left(\frac{3}{5}\right) - \frac{3}{5}r^2$ $= 0.2044r^2$		

THREE (a)(i)	Since $y = k + \frac{1}{m}\ln\left(\frac{k}{x}\right)$ $\frac{k}{x} = \mathrm{e}^{m(y-k)}, \; x = k\mathrm{e}^{-m(y-k)}$ So the area A is given by $A = \int_0^a k\mathrm{e}^{m(k-y)}\mathrm{d}y = k\mathrm{e}^{mk}\int_0^a \mathrm{e}^{-my}\mathrm{d}y$ $= k\mathrm{e}^{mk}\left[-\frac{1}{m}\mathrm{e}^{-my}\right]_0^a = -\frac{k}{m}\mathrm{e}^{mk}\left(\mathrm{e}^{-ma} - 1\right)$ When $-\frac{k}{m}\mathrm{e}^{mk}\left(\mathrm{e}^{-ma} - 1\right) = \frac{p}{100}\bullet\frac{k}{m}\mathrm{e}^{mk}$	**6** **4**	
	$-\left(\mathrm{e}^{-ma} - 1\right) = \frac{p}{100} \quad \left(\frac{k}{m}\mathrm{e}^{mk} \neq 0\right)$ $\mathrm{e}^{ma} = \frac{100}{100 - p}$ $a = \frac{1}{m}\ln\left(\frac{100}{100 - p}\right).$		Or equivalent.

THREE (a)(ii)	$V = \pi\int_0^a k^2 e^{-2m(y-k)}dy$ $V = \pi k^2 \int_0^a e^{-2m(y-k)}dy = \pi k^2\left[-\frac{1}{2m}\left(e^{-2m(y-k)}\right)\right]_0^a$ $= -\frac{\pi k^2}{2m}\left(e^{-2m(a-k)} - e^{2mk}\right)$ $= -\frac{\pi k^2}{2m}e^{2mk}\left(e^{-2ma} - 1\right)$ So if $\frac{V}{A} \le e^{mk}$ $\frac{-\frac{\pi k^2}{2m}e^{2mk}\left(e^{-2ma} - 1\right)}{-\frac{k}{m}e^{mk}\left(e^{-ma} - 1\right)} \le e^{mk}$ $\frac{\pi k\left(e^{-2ma} - 1\right)}{2\left(e^{-ma} - 1\right)} \le 1$	**8** **6**	If not simplified: –1 mark.
	$\frac{\pi k\left(e^{-ma} + 1\right)\left(e^{-ma} - 1\right)}{2\left(e^{-ma} - 1\right)} \le 1 \quad \left(e^{-ma} \ne 1\right)$ $\pi k\left(e^{-ma} + 1\right) \le 2$ $e^{-ma} \le \frac{2}{\pi k} - 1$ $a \ge -\frac{1}{m}\ln\left(\frac{2}{\pi k} - 1\right) = \frac{1}{m}\ln\left(\frac{\pi k}{2 - \pi k}\right).$ **Method 2** $\frac{\pi k\left(e^{-2ma} - 1\right)}{2\left(e^{-ma} - 1\right)} \le 1,$ $\pi k\left(e^{-2ma} - 1\right) \ge 2\left(e^{-ma} - 1\right)$ since $e^{-ma} - 1 < 0$ $\pi k e^{-2ma} - 2e^{-ma} + 2 - \pi k \ge 0,$ and since $e^{ma} > 0$ $\left(2 - \pi k\right)e^{2ma} - 2e^{ma} + \pi k \ge 0$ $\left(e^{ma} - 1\right)\left(\left(2 - \pi k\right)e^{ma} - \pi k\right) \ge 0$ but $e^{ma} > 1$ for $ma > 0$, so $\left(2 - \pi k\right)e^{ma} - \pi k \ge 0$ $e^{ma} \ge \frac{\pi k}{2 - \pi k}$ and $a \ge \frac{1}{m}\ln\left(\frac{\pi k}{2 - \pi k}\right)$		Or equivalent.

THREE (b) — 8

$$y = x\sin nx + \frac{1}{n}\cos nx$$

$$\frac{dy}{dx} = \sin nx + nx\cos nx - \sin nx$$

$$= nx\cos nx$$

$$I_n - I_{n-1}$$

$$= \int_0^{\pi}\left(\tfrac{1}{2}\pi - x\right)\sin\left(n+\tfrac{1}{2}\right)x\ \mathrm{cosec}\tfrac{1}{2}x - \left(\tfrac{1}{2}\pi - x\right)\sin\left(n-\tfrac{1}{2}\right)x\,\mathrm{cosec}\tfrac{1}{2}x\,dx$$

$$= \int_0^{\pi}\left(\tfrac{1}{2}\pi - x\right)\mathrm{cosec}\tfrac{1}{2}x\left(\sin\left(n+\tfrac{1}{2}\right)x - \sin\left(n-\tfrac{1}{2}\right)x\right)dx$$

$$= \int_0^{\pi}\left(\tfrac{1}{2}\pi - x\right)\mathrm{cosec}\tfrac{1}{2}x\left(2\cos nx\sin\tfrac{1}{2}x\right)dx$$

$$= \int_0^{\pi}\left(\tfrac{1}{2}\pi - x\right)2\cos nx\,dx$$

$$= \int_0^{\pi}\pi\cos nx - 2x\cos nx\,dx$$

$$= \left[\frac{\pi}{n}\sin nx - \frac{2}{n^2}\left(nx\sin nx + \cos nx\right)\right]_0^{\pi} \qquad \text{using the first result}$$

$$= \left(\frac{\pi}{n}\sin n\pi - \frac{2}{n^2}\left(n\pi\sin n\pi + \cos n\pi\right)\right) - \left(\frac{\pi}{n}\sin 0 - \frac{2}{n^2}\left(n\pi\sin 0 + \cos 0\right)\right)$$

6

$$= \left(0 - \frac{2}{n^2}\left(0+1\right)\right) - \left(0 - \frac{2}{n^2}\left(0+1\right)\right) \text{ for } n \text{ even}$$

$$= 0 \text{ for } n \text{ even}$$

$$= \left(0 - \frac{2}{n^2}\left(0-1\right)\right) - \left(0 - \frac{2}{n^2}\left(0+1\right)\right) \text{ for } n \text{ odd}$$

$$= \frac{2}{n^2} + \frac{2}{n^2}$$

$$= \frac{4}{n^2} \text{ for } n \text{ an odd number.}$$

$I_0 = 0$ (given)

$$n = 1\,(n \text{ odd}),\quad I_1 - I_0 = \frac{4}{1^2} \quad \text{so} \quad I_1 = \frac{4}{1^2}$$

$$n = 2\,(n \text{ even}),\quad I_2 - I_1 = 0 \quad \text{so} \quad I_2 = I_1 = \frac{4}{1^2}$$

$$n = 3\,(n \text{ odd}),\quad I_3 - I_2 = \frac{4}{3^2} \quad \text{so} \quad I_3 = \frac{4}{3^2} + \frac{4}{1^2}$$

Three (b) contd	$n = 4\,(n \text{ even}),\quad I_4 - I_3 = 0 \quad \text{so } I_4 = I_3 = \frac{4}{3^2} + \frac{4}{1^2}$ $n \text{ even},\ I_n = \frac{4}{1^2} + \frac{4}{3^2} + \ldots\ldots.. + \frac{4}{(n-1)^2} = 4\sum_{2}^{n} \frac{1}{(\mathrm{i}-1)^2}$ $n \text{ odd},\ I_n = \frac{4}{1^2} + \frac{4}{3^2} + \ldots\ldots.. + \frac{4}{n^2} = 4\sum_{1}^{n} \frac{1}{\mathrm{i}^2}$		$\sum$ notation not required.

FOUR (a)(i)	**Method 1** $$\frac{dy}{dx} = \frac{dy}{dt} \bullet \frac{dt}{dx} = \frac{g'(t)}{f'(t)}$$ $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{f'(t)g''(t) - g'(t)f''(t)}{\left[f'(t)\right]^2} \bullet \frac{1}{f'(t)} = 0$$ then $$f'(t)g''(t) - g'(t)f''(t) = 0$$ $$f'(t)g''(t) = g'(t)f''(t)$$ $$\frac{dx}{dt} \bullet \frac{d^2y}{dt^2} = \frac{dy}{dt} \bullet \frac{d^2x}{dt^2}$$ as required. **Method 2** $\frac{d^2y}{dx^2} = 0$ so integrating wrt x $\frac{dy}{dx} = k$ k constant, and integrating again wrt x $y = kx + c$ c constant Differentiating this wrt t twice $\frac{dy}{dt} = k\frac{dx}{dt}$ and then $\frac{d^2y}{dt^2} = k\frac{d^2x}{dt^2}$ Hence $\frac{dx}{dt} \bullet \frac{d^2y}{dt^2} = k\frac{dx}{dt} \bullet \frac{d^2x}{dt^2}$ but $k = \frac{dy}{dx}$, so $\frac{dx}{dt} \bullet \frac{d^2y}{dt^2} = \frac{dy}{dx} \bullet \frac{dx}{dt} \bullet \frac{d^2x}{dt^2}$	**6**	$\frac{1}{f'(t)}$ must be present. Accept function notation.

and $$\frac{dx}{dt}\bullet\frac{d^2y}{dt^2}=\frac{dy}{dt}\bullet\frac{d^2x}{dt^2}$$

Method 3

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$=\frac{d}{dx}\left(\frac{dy}{dt}\bullet\frac{dt}{dx}\right)$$

$$=\frac{d}{dx}\left(\frac{dy}{dt}\right)\frac{dt}{dx}+\frac{d}{dx}\left(\frac{dt}{dx}\right)\frac{dy}{dt}$$

$$=\frac{d}{dt}\left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right)^2+\frac{d^2t}{dx^2}\bullet\frac{dy}{dt}$$

$$=\frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2+\frac{d^2t}{dx^2}\bullet\frac{dy}{dt}=0$$

but $$\frac{dt}{dx}=\left(\frac{dx}{dt}\right)^{-1}$$

$$\frac{d^2t}{dx^2}=-\left(\frac{dx}{dt}\right)^{-2}\frac{d^2x}{dt^2}\bullet\frac{dt}{dx}$$

$$=-\frac{d^2x}{dt^2}\left(\frac{dt}{dx}\right)^3$$

hence $$\frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2=-\left(-\frac{d^2x}{dt^2}\left(\frac{dt}{dx}\right)^3\right)\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2}\bullet\frac{dx}{dt}=\frac{d^2x}{dt^2}\bullet\frac{dy}{dt}$$

Method 4

$\frac{d^2y}{dx^2}=0$ so integrating wrt x

$\frac{dy}{dx}=k=\frac{dy}{dt}\bullet\frac{dt}{dx}$ and so $\frac{dy}{dt}=k\frac{dx}{dt}$

Differentiating this wrt t

$\frac{d^2y}{dt^2}=k\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}=\frac{dy}{dt}\bullet\frac{dt}{dx}\bullet\frac{d^2x}{dt^2}$ so

$$\frac{d^2y}{dt^2}\bullet\frac{dx}{dt}=\frac{d^2x}{dt^2}\bullet\frac{dy}{dt}$$

FOUR (a)(ii)	$x = a\cos t + \frac{1}{2}b\cos 2t$ $\frac{dx}{dt} = -a\sin t - b\sin 2t$ $\frac{d^2x}{dt^2} = -a\cos t - 2b\cos 2t$ $y = a\sin t + \frac{1}{2}b\ \sin 2t$ $\frac{dy}{dt} = a\cos t + b\cos 2t$ $\frac{d^2y}{dt^2} = -a\sin t - 2b\sin 2t$ For points of inflection $\frac{d^2y}{dx^2} = 0$, and using the result from 4(a)(i) $\left(-a\sin t - b\sin 2t\right)\left(-a\sin t - 2b\sin 2t\right)$ $-\left(a\cos t + b\cos 2t\right)\left(-a\cos t - 2b\cos 2t\right) = 0$ $a^2\sin^2 t + 2ab\sin t\sin 2t + ab\sin t\sin 2t + 2b^2\sin^2 2t +$ $a^2\cos^2 t + 2ab\cos t\cos 2t + ab\cos t\cos 2t + 2b^2\cos^2 2t = 0$ $a^2\left(\sin^2 t + \cos^2 t\right) + 3ab\left(\cos t\cos 2t + \sin t\sin 2t\right) +$ $2b^2\left(\sin^2 2t + \cos^2 2t\right) = 0$ $a^2 + 3ab\cos t + 2b^2 = 0$ $\cos t = \frac{-\left(a^2 + 2b^2\right)}{3ab}$	**6**	
FOUR (b)	$\frac{dx}{dy} \bullet \frac{d^2y}{dx^2} = k\frac{dy}{dx}$ so $\frac{d\left(\frac{dy}{dx}\right)}{dx} = k\left(\frac{dy}{dx}\right)^2$ Let $z = \frac{dy}{dx}$ $\frac{dz}{dx} = kz^2$ $\int \frac{1}{z^2}\,dz = k\int dx$ $-\frac{1}{z} = kx + C$ but $z = 1$ when $x = 0$ (given), so $C = -1$	**8**	

FOUR (b) contd	$-\frac{1}{z} = kx - 1$ $\frac{dy}{dx} = \frac{1}{1-kx}$ $y = \int \frac{1}{1-kx}\,dx = -\frac{1}{k}\ln\lvert 1-kx\rvert + C$ but when $x = 0$, $y = 1$ (given) so $C = 1$ $y = 1 - \frac{1}{k}\ln\lvert 1-kx\rvert$	**6**	
	and when $y = 2$, $1 = -\frac{1}{k}\ln\lvert 1-kx\rvert$, $1 - kx = e^{-k}$, $x = \frac{1}{k}\left(1 - e^{-k}\right)$ so when $y = 2$, $\frac{dy}{dx} = \frac{1}{1-kx} = \frac{1}{e^{-k}} = e^{k}$. **Or** Let $\frac{dy}{dx} = p$, then $\frac{d^2y}{dx^2} = \frac{dp}{dy}p$ and so for $\frac{dx}{dy} \bullet \frac{d^2y}{dx^2} = k\frac{dy}{dx}$ $p\frac{dp}{dy} = kp^2$ $\int \frac{1}{p}dp = k\int dy$ and $\ln\lvert p\rvert = ky + C$ $p = Ae^{ky}$ but when $y = 1$, $p = \frac{dy}{dx} = 1$, so $A = e^{-k}$ $p = e^{k(y-1)}$ and so when $y = 2$ $p = \frac{dy}{dx} = e^{k}$		

FIVE (a)	$\cos(2A+B)=\cos 2A\cos B-\sin 2A\sin B$ $=(2\cos^2 A-1)\cos B-2\sin A\cos A\sin B$ $A=B=\theta$ then $\frac{1}{4}\cos 3\theta=\frac{1}{4}[\cos 2\theta\cos\theta-\sin 2\theta\sin\theta]$ $=\frac{1}{4}\left[(2\cos^2\theta-1)\cos\theta-2\sin\theta\cos\theta\sin\theta\right]$ $=\frac{1}{4}\left[2\cos^3\theta-\cos\theta-2\cos\theta(1-\cos^2\theta)\right]$ $=\frac{1}{4}\left[4\cos^3\theta-3\cos\theta\right]$ $=\cos^3\theta-\frac{3}{4}\cos\theta$	**8** **2**	Marks of 2, 4, and 2 for the parts may be given independently. Only one minor error (ME) allowed.
	$27x^3-9x=1$ Let $x=\frac{2}{3}\cos\theta$ $27\left(\frac{2}{3}\cos\theta\right)^3-9\left(\frac{2}{3}\cos\theta\right)=1$ $8\cos^3\theta-6\cos\theta=1$ $\cos^3\theta-\frac{6}{8}\cos\theta=\frac{1}{8}$ $\cos^3\theta-\frac{3}{4}\cos\theta=\frac{1}{8}$ $\frac{1}{4}\cos 3\theta=\frac{1}{8}$ $\cos 3\theta=\frac{1}{2}$ $3\theta=\frac{\pi}{3},\ 2\pi-\frac{\pi}{3},\ 2\pi+\frac{\pi}{3},\ \ldots$ $\theta=\frac{\pi}{9},\ \frac{5\pi}{9},\ \frac{7\pi}{9}$	**6**	Accept use of general formula. Decimals: –1 mark. One solution only, ME, 5 marks Hence must be used.
	The roots of the equation are $x=\frac{2}{3}\cos\frac{\pi}{9};\ \frac{2}{3}\cos\frac{5\pi}{9};\ \frac{2}{3}\cos\frac{7\pi}{9}.$ The product $\cos\frac{\pi}{9}\cos\frac{3\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}=\frac{1}{2}\left(\frac{1}{27}\right)\left(\frac{3}{2}\right)^3=\frac{1}{16}.$		Accept decimal 0.0625 only if fractions present.

FIVE (b)	$-\frac{1}{2k}+\frac{3}{k+1}-\frac{5}{2(k+2)} = \frac{-(k+1)(k+2)+6k(k+2)-5k(k+1)}{2k(k+1)(k+2)}$ $= \frac{-(k^2+3k+2)+6k^2+12k-5k^2-5k}{2k(k+1)(k+2)}$ $= \frac{2(2k-1)}{2k(k+1)(k+2)} = \frac{2k-1}{k(k+1)(k+2)}$ so $f(n)=\sum_{k=1}^{k=n}\left(-\frac{1}{2k}+\frac{3}{k+1}-\frac{5}{2(k+2)}\right)$.	**8** **4**	
	$f(n) = -\frac{1}{2}+\frac{3}{2}-\cancel{\frac{5}{6}}$ $-\frac{1}{4}+\cancel{\frac{3}{3}}-\cancel{\frac{5}{8}}$ $-\cancel{\frac{1}{6}}+\cancel{\frac{3}{4}}-\frac{5}{10}$ $\ldots$ $-\cancel{\frac{1}{2(n-1)}}+\cancel{\frac{3}{n}}-\frac{5}{2(n+1)}$ $-\cancel{\frac{1}{2n}}+\frac{3}{n+1}-\frac{5}{2(n+2)}$ Terms cancel out in threes, leaving just 6. $f(n) = -\frac{1}{2}+\frac{3}{2}-\frac{1}{4}-\frac{5}{2(n+1)}+\frac{3}{n+1}-\frac{5}{2(n+2)}$ $f(n) = \frac{3}{4}+\frac{1}{2(n+1)}-\frac{5}{2(n+2)}$ so as $n \to \infty$, $f(n) \to \frac{3}{4}$.		Cancelling not required. Or equivalent alternative method.
SIX (a)	$4x^2-y^2-16hx+2hy+15h^2-4a^2=0$ $4(x^2-4hx)-(y^2-2hy)=4a^2-15h^2$ $4(x-2h)^2-(y-h)^2=4a^2-15h^2+16h^2-h^2=4a^2$ and $\frac{(x-2h)^2}{a^2}-\frac{(y-h)^2}{4a^2}=1$ and this is a hyperbola centre $(2h,h)$.	**6** **2**	Marks of 2, 2, and 2 for the parts may be given independently. Only one minor (ME) allowed.
	Since $e^2=1+\frac{b^2}{a^2}$, $e^2-1=\frac{4a^2}{a^2}=4$. So the given line is $y=4x$, with gradient 4.		

SIX (a) Contd	$\frac{2(x-2h)}{a^2} - \frac{2(y-h)}{4a^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4(x-2h)}{y-h}$	**2**	Or differentiating the original form. Gradient must be correct.
	and when $x = p, y = q$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4\left(p-2h\right)}{q-h}$ and so $\frac{4\left(p-2h\right)}{q-h} = 4$ $p - 2h = q - h,\ p - q = h.$	**2**	
SIX (b)	$\mathrm{G} = (-ae,0) = (-a\sqrt{5},0)$; $\mathrm{F} = (ae,0) = (a\sqrt{5},0)$ for a standard hyperbola, so **Either:** We can work with these points and then translate the line by $\begin{pmatrix}2h\\h\end{pmatrix}$ For A: $\frac{\left(a\sqrt{5}\right)^2}{a^2} \bullet \frac{y^2}{4a^2} = 1,\ \frac{y^2}{4a^2} = 5 - 1 = 4,\ y^2 = 16a^2,\ y = \pm 4a$ So $\mathrm{A} = (a\sqrt{5},4a)$ and the line AG is: $y - 0 = \frac{4a}{a\sqrt{5} + a\sqrt{5}}\left(x + a\sqrt{5}\right)$ $y = \frac{2}{\sqrt{5}}\left(x + a\sqrt{5}\right)$	**8** **4**	
	then translating the line by $\begin{pmatrix}2h\\h\end{pmatrix}$ $y - h = \frac{2}{\sqrt{5}}\left(x - 2h + a\sqrt{5}\right)$ **or** $\sqrt{5}y = 2x + \left(\sqrt{5} - 4\right)h + 2a\sqrt{5}$. **Or:** with a translation $\begin{pmatrix}2h\\h\end{pmatrix}$ of the points first we get $\mathrm{G} = (2h - a\sqrt{5},h)$, $\mathrm{F} = (2h + a\sqrt{5},h)$ Hence for A $\frac{(2h + a\sqrt{5} - 2h)^2}{a^2} - \frac{(y-h)^2}{4a^2} = 1$		Or equivalent.

SIX (b) Contd	$\frac{(y-h)^2}{4a^2} = 5 - 1 = 4$ $y - h = \sqrt{16a^2} = 4a$ and $y = 4a + h$. $\text{A} = (2h + a\sqrt{5}, 4a + h)$.	**4**	For both A and G.
	Hence the equation of AG is given by: $y - h = \frac{4a + h - h}{2h + a\sqrt{5} - \left(2h - a\sqrt{5}\right)}\left(x - \left(2h - a\sqrt{5}\right)\right)$ $y - h = \frac{4a}{2a\sqrt{5}}\left(x - \left(2h - a\sqrt{5}\right)\right) = \frac{2}{\sqrt{5}}\left(x - \left(2h - a\sqrt{5}\right)\right)$ $\sqrt{5}y = 2x + \left(\sqrt{5} - 4\right)h + 2a\sqrt{5}$.		Accept with simplified gradient $\frac{2}{\sqrt{5}}$. Accept decimals.
SIX (c)	Since $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ gives $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b^2x}{a^2y}$, $m = \frac{b^2x_1}{a^2y_1}$, $x_1 = \frac{ma^2}{b^2}y_1$ But $\frac{x_1{}^2}{a^2} - \frac{y_1{}^2}{b^2} = 1$ and so $\frac{1}{a^2}\left(\frac{ma^2y_1}{b^2}\right)^2 - \frac{y_1{}^2}{b^2} = 1$ $y_1{}^2\left(m^2a^2 - b^2\right) = b^4$ $y_1 = \pm\frac{b^2}{\sqrt{m^2a^2 - b^2}}$	**8** **6**	Accept the alternative $x_1 = \frac{\pm ma^2}{\sqrt{m^2a^2 - b^2}}$
	Since $y = mx - \frac{b^2}{y_1}$ $y = mx \pm \sqrt{m^2a^2 - b^2}$ **Or:** in $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ $\frac{x\frac{ma^2}{b^2}y_1}{a^2} - \frac{yy_1}{b^2} = 1$ or $y_1\left(\frac{mx}{b^2} - \frac{y}{b^2}\right) = 1$	**6**	

SIX (c) contd	$(mx - y) = \dfrac{b^2}{y_1} = \dfrac{b^2}{\pm\dfrac{b^2}{\sqrt{m^2a^2 - b^2}}} = \pm\sqrt{m^2a^2 - b^2}$ and $y = mx \pm \sqrt{m^2a^2 - b^2}$.		
SIX (d)	This question is equivalent to the problem obtained by translating the hyperbola and using $\dfrac{x_1^{\ 2}}{a^2} - \dfrac{y_1^{\ 2}}{4a^2} = 1$ and a tangent through the point $\left(\dfrac{2a}{3}, 0\right)$. Using the answer to part (c) and the point $\left(\dfrac{2a}{3}, 0\right)$ $0 = \dfrac{2am}{3} \pm \sqrt{m^2a^2 - 4a^2}$ $\dfrac{4a^2m^2}{9} = (m^2 - 4)a^2$ $5m^2 = 36$, $m = \pm\dfrac{6}{\sqrt{5}}$ and we want the + sign as asked so $m = \dfrac{6}{\sqrt{5}}$. The asymptote is $y = \dfrac{bx}{a} = \dfrac{2ax}{a} = 2x$, with gradient 2.	**8** **6**	
	So the angle α between the two lines is given by $\tan\alpha = \dfrac{m_1 - m_2}{1 + m_1m_2} = \dfrac{\dfrac{6}{\sqrt{5}} - 2}{1 + 2.\dfrac{6}{\sqrt{5}}} = \dfrac{6 - 2\sqrt{5}}{\sqrt{5} + 12}$ $= \dfrac{(6 - 2\sqrt{5})(\sqrt{5} - 12)}{(\sqrt{5} + 12)(\sqrt{5} - 12)} = \dfrac{30\sqrt{5} - 82}{-139} = \dfrac{82 - 30\sqrt{5}}{139}$ and $\alpha = \tan^{-1}\left(\dfrac{2(41 - 15\sqrt{5})}{139}\right)$.		

An aggregate mark of 120 from six questions was used in Scholarship Calculus. In 2005, candidates who achieved 47–120 were awarded outstanding scholarship, and candidates who achieved 20–46 were awarded scholarship.